Deformed Heisenberg algebras, a Fock-space representation and the Calogero model

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Abstract. We describe generally deformed Heisenberg algebras in one dimension. The condition for a generalized Leibniz rule is obtained and solved. We analyze conditions under which deformed quantum-mechanical problems have a Fock-space representation. One solution of these conditions leads to a q-deformed oscillator already studied by Lorek et al., and reduces to the harmonic oscillator only in the infinite-momentum frame. The other solution leads to the Calogero model in ordinary quantum mechanics, but reduces to the harmonic oscillator in the absence of deformation.

1 Introduction

In the last few years considerable attention has been given to quantum groups acting on noncommutative spaces [1]. They represent a generalization of the concept of symmetries. A number of papers have established a connection between quantum groups, spaces, and q-deformed physics. The main idea behind these connections is that the geometrical coordinates obey generalized (braid) statistics. A specific approach to q-deformed phase space has been introduced by Schwenk and Wess [2] and Fichtmüller et al. [3]. In the light of these ideas, the q-deformed harmonic oscillator has been studied [4]. Alternatively, the q-deformed harmonic oscillator based on an algebra of q-deformed creation and annihilation operators has been studied by Biedenharn and Macfarlane [5,6]. A unified view of deformed single-mode oscillator algebras has been proposed [7,8].

A systematic approach to q-deformed Heisenberg algebras in one dimension and higher dimensions has been developed and applied to gauge field theories on noncommutative spaces [9,10]. As the simplest example of noncommutative structure, the one-dimensional q-deformed Heisenberg algebra has been considered [2]. In addition, a differential calculus entirely based on the algebra has been derived. The laws of physics based on this calculus have been formulated. The representation of the algebra has been studied in the adapted quantum-mechanical scheme.

In this paper we generalize the one-dimensional Heisenberg algebra in such a way that the momentum in the x-representation is given by the function depending on one parameter or more parameters. General conditions on this function are imposed and, specially, the condition for a generalized Leibniz rule is obtained and solved. Our main goal is to investigate under which condition the quantum-mechanical problem (deformed and undeformed) can be presented in the Fock-space representation. For q-deformed quantum mechanics there is only one solution constructed by Lorek et al. [4], satisfying the Fock-space representation condition. However, we point out that this oscillator reduces to the harmonic one in the infinite-momentum frame.

We have shown that there is only one solution satisfying the Fock-space representation condition in generally deformed quantum mechanics (approaching the simple oscillator in ordinary mechanics). This solution can be interpreted as the Calogero model in the harmonic potential in ordinary quantum mechanics. Alternatively, the Calogero model can be viewed as deformed harmonic oscillator.

The paper is organized as follows. In Sect. 2 we describe generally deformed Heisenberg algebras in one dimension and discuss the generalized Leibniz rule. In Sect. 3 we analyze conditions under which deformed quantum-mechanical problems have a Fock-space representation. In Sect. 4 we find that the Calogero model can be interpreted as a deformed Heisenberg algebra with a Fock-space representation. Finally, in Sect. 5 we summarize the main result of this paper.

2 Deformed Heisenberg algebras in one dimension

In a series of papers [2–4, 10–13] a formal calculus entirely based on an algebra has been developed. Especially, the q-deformed Heisenberg algebra has been considered. The model is based on the q-deformed relations

$$q^{1/2}xp - q^{-1/2}px = \mathrm{i}\Lambda,$$
$$\Lambda p = qp\Lambda,$$

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$$\Lambda x = q^{-1}x\Lambda, \ q \in \mathbf{R}, \ q \neq 0. \tag{1}$$

Furthermore, the algebra has to be a star algebra to allow a physical interpretation. The element x of the algebra will be identified with the observable for position in space, the element p with the canonical conjugate observable (called momentum). Observables have to be represented by self-adjoint linear operators in Hilbert space. This will guarantee real eigenvalues and a complete set of orthogonal eigenvectors. Hence, the requirement is

$$\bar{x} = x, \ \bar{p} = p.$$

Then, for $q \in \mathbf{R}, \ q \neq 1$,

$$px = \mathrm{i}(q - q^{-1})^{-1}\left(q^{-1/2}\Lambda - q^{1/2}\bar{\Lambda}\right),$$
$$xp = \mathrm{i}(q - q^{-1})^{-1}\left(q^{1/2}\Lambda - q^{-1/2}\bar{\Lambda}\right). \tag{2}$$

Furthermore, the algebra is extended by $\Lambda^{-1}, \ x^{-1}$, and it is demanded that

$$\bar{\Lambda} = \Lambda^{-1}.$$

A field f is defined as an element of the subalgebra generated by x and x^{-1}, and then completed by a formal power series $f(x) \in \mathcal{A}_x$. The above algebra $\mathcal{A}$, (1), can be represented on $\mathcal{A}_x$ in a natural way (the x-representation in deformed quantum mechanics):

$$x \to x, \ p \to -\mathrm{i}\nabla,$$
$$N = x\frac{\mathrm{d}}{\mathrm{d}x}, \ \bar{N} = -\frac{\mathrm{d}}{\mathrm{d}x}x = -1 - N, \ Nx^n = nx^n,$$
$$\Lambda = q^{-N-1/2}. \tag{3}$$

Then

$$\bar{\Lambda} = q^{-\bar{N}-1/2} = q^{N+1/2} = \Lambda^{-1},$$
$$x\nabla = \frac{q^N - q^{-N}}{q - q^{-1}} = \frac{\sinh(Nh)}{\sinh(h)}, \ q = \mathrm{e}^h,$$
$$\nabla x = \frac{\sinh((N+1)h)}{\sinh(h)}. \tag{4}$$

Let us generalize the above algebra and consider its representation in the following way:

$$x\nabla = f_h(N),$$
$$\nabla x = -\overline{f_h(N)} = -\bar{f}_h(\bar{N}) = -\bar{f}_h(-N-1) = f_h(N+1),$$
$$\text{if } \bar{f}_h(-N) = -f_h(N). \tag{5}$$

Then

$$\nabla = \frac{1}{x}f_h(N).$$

We consider $f_h(x)$ as a real function, with the restriction $f_0(N) = N$ as the continuous deformation parameters $h \to 0$. One easily finds

$$\nabla x^n = f(n)x^{n-1}, \ n \in \mathbf{Z}, \ f(1) = 1, \ f(0) = 0.$$

For example,

$$\nabla x = 1, \ \nabla c = 0, \ \nabla f(x) = \nabla \sum_k c_k x^k = \sum_k c_k f(k)x^{k-1}.$$

Next, we consider what conditions the function f should satisfy in order that a generalized derivative should satisfy the generalized Leibniz rule [10]:

$$\nabla(x^{n+m}) = (\nabla x^n)(\varphi(m)x^m) + (\varphi(-n)x^n)(\nabla x^m)$$
$$\implies f(n+m) = f(n)\varphi(m) + \varphi(-n)f(m). \tag{6}$$

The functions $f(t), \ \varphi(t)$ should satisfy the following conditions in the undeformed case:

$$\lim_{h \to 0} \varphi_h(t) = 1, \ \forall t,$$
$$\lim_{h \to 0} f_h(t) = t.$$

These conditions suggest a simple ansatz, $\varphi_h(t) = cf'_h(t)$, i.e.,

$$\varphi_h(-t) = \varphi_h(t),$$
$$f(t+u) = cf(t)f'(u) + cf'(t)f(u),$$
$$f(2t) = 2cf(t)f'(t), \ cf'(0) = 1. \tag{7}$$

The general solution is given by two one-parameter families:

$$\text{(i)} \ \ f_h(t) = \frac{\sinh(th)}{\sinh(h)}, \ \varphi_h(0) = \frac{ch}{\sinh(h)} = 1,$$

considered by Wess et al. [3,4,10] (connected with the quantum-group consideration for $q \in \mathbb{R}, \ q = \exp(h)$), and

$$\text{(ii)} \ \ f_h(t) = \frac{\sin(th)}{\sin(h)}, \ \varphi_h(t) = \cos(th).$$

To our knowledge, this latter family has not been considered and might be connected with the quantum group for $q \in \mathbf{S_1}, \ q = \exp(\mathrm{i}h), \ h \in \mathbb{R}$.

Remark 1. There are other solutions not covered by the ansatz (7). For example,

$$\varphi(t) = \begin{cases} q^t \\ q^{-t} \\ \frac{q^t + q^{-t}}{2} = \cosh(th) \end{cases} \quad \text{for } f_h(t) = \frac{\sinh(th)}{\sinh(h)}$$

and

$$\varphi(t) = \begin{cases} q^{\mathrm{i}t} \\ q^{-\mathrm{i}t} \\ \frac{q^{\mathrm{i}t} + q^{-\mathrm{i}t}}{2} = \cos(th) \end{cases} \quad \text{for } f_h(t) = \frac{\sin(th)}{\sin(h)}.$$

Other solutions of the initial Leibniz condition are not known to us at present.

Remark 2. For $q \in S_1, \ q = \mathrm{e}^{\mathrm{i}h}, \ h \in \mathbb{R}$, the corresponding algebra is

$$q^{1/2}xp - q^{-1/2}px = \mathrm{i}\Lambda,$$
$$\Lambda p = qp\Lambda, \ \Lambda x = q^{-1}x\Lambda,$$
$$\bar{x} = x, \ \bar{p} = p, \ \bar{\Lambda} = \Lambda = q^{-N-1/2},$$
$$\nabla = \frac{1}{x}\frac{\sin(Nh)}{\sin(h)}, \ \sin(h) \neq 0.$$

The general function $f(t)$, with the conditions $\bar{f} = f$, $f(t)$−odd, can be written as

$$f(N) = \sum_{k=0}^{\infty} c_k N^{2k+1}. \qquad (8)$$

Now, we define the Hamiltonian in one dimension as

$$H = \frac{1}{2}p^2 + V(x) = -\frac{1}{2}\nabla^2 + V(x),$$
$$H\Psi = i\frac{\partial}{\partial t}\Psi = E\Psi.$$

The simplest examples of dynamics are given by the following cases:

(a)The free Hamiltonian with $V(x) = 0$,

$$\Psi(x) \sim \left[e^{ipx}\right] \equiv \sum_{n=0}^{\infty} \frac{(ipx)^n}{[f(n)]!},$$
$$[f(n)]! = f(n)\cdots f(2)f(1),$$
$$\nabla\left[e^{ipx}\right]_f = ip\left[e^{ipx}\right]_f, \; p \in \mathbb{R}. \qquad (9)$$

(b) The quadratic potential $V(x) = x^2$.

Remark 3. Orthogonality, completeness, and q-deformed integrals can be considered analogously as in [10].

3 Fock-space representation and one-dimensional quantum-mechanical problems

We consider a class of deformed quantum-mechanical problems in one dimension which allow the Fock-space representation of a deformed single (one-dimensional) oscillator. The Hamiltonian is given by

$$H = \frac{1}{2}p^2 + V(x) = -\frac{1}{2}\nabla^2 + V(x),$$
$$\nabla = \frac{1}{x}f(N), \; N = x\frac{d}{dx}. \qquad (10)$$

We assume that the spectrum is discrete and bounded, i.e., that there exists a ground state with the lowest energy E_0:

$$H\Psi_n = E_n\Psi_n, \; n \in N_0.$$

Then we pose the question which class of potentials $V(x)$ (for the fixed deformation $f(N)$) leads to the deformed oscillator with the Fock-space representation in the following way:

$$H = a^\dagger a + E_0,$$
$$a^\dagger a = E(N) - E_0,$$
$$[N, a] = -a, \; [N, a^\dagger] = a^\dagger,$$
$$a = F(p, x, N), \; \Psi_n = (a^\dagger)^n|0\rangle, \; \Psi_0 = |0\rangle,$$
$$a|0\rangle = a\Psi_0 = 0, \qquad (11)$$

where $a^\dagger$ is the Hermitian conjugate of a, $a^\dagger \neq a$.

Remark 4. There is a class of problems which can be mapped onto the oscillator problem [14, 15]. However, here we insist on the Fock-space representation.

If

$$|n\rangle = (a^\dagger)^n|0\rangle, \; n \in N_0, \; \langle n|n\rangle > 0, \; \forall n \in N_0,$$
$$a^\dagger a = E(N),$$

and

$$a^\dagger a(a^\dagger)^n|0\rangle = E(n)(a^\dagger)^n|0\rangle,$$

then

$$a(a^\dagger)^n|0\rangle = E(n)(a^\dagger)^{n-1}|0\rangle + |v\rangle, \; a^\dagger|v\rangle = 0.$$

This would imply that for some p

$$(a^\dagger)^{p-1}|0\rangle \neq 0, \; \text{and} \; (a^\dagger)^p|0\rangle = 0,$$

in contradiction with our initial assumption that $|n\rangle \neq 0, \; \forall n \in N_0$. Hence

$$a(a^\dagger)^n|0\rangle = E(n)(a^\dagger)^{n-1}|0\rangle,$$

i.e.,

$$aa^\dagger = E(N + 1).$$

Hence, the two Hermitian operators $a^\dagger a$ and $aa^\dagger$ commute:

$$[aa^\dagger, a^\dagger a] = 0. \qquad (12)$$

The opposite is also true. If $aa^\dagger \neq a^\dagger a$, $[aa^\dagger, a^\dagger a] = 0$, and $a|0\rangle = 0$, then we simultaneously diagonalize the Hermitian operators $a^\dagger a$ and $aa^\dagger$. Assuming the discrete spectrum, we have

$$a^\dagger a|n\rangle = \varphi(n)|n\rangle,$$
$$aa^\dagger|n\rangle = \psi(n)|n\rangle, \; n \in N_0, \qquad (13)$$

with $\varphi(n)$, $\psi(n)$ bounded from below. Starting from any state $|k\rangle$, we have

$$a^\dagger a|k\rangle = \varphi_k|k\rangle,$$
$$aa^\dagger|k\rangle = \psi_k|k\rangle. \qquad (14)$$

Then

$$a^\dagger a(a^\dagger|k\rangle) = a^\dagger \psi_k|k\rangle = \varphi(k')a^\dagger|k\rangle.$$

Since $[aa^\dagger, a^\dagger a] = 0$, we have

$$aa^\dagger(a^\dagger|k\rangle) = \psi(k')a^\dagger|k\rangle.$$

In the same way we find

$$a^\dagger a(a^\dagger a^\dagger|k\rangle) = a^\dagger \psi(k')a^\dagger|k\rangle = \varphi(k'')(a^\dagger a^\dagger)|k\rangle,$$

and

$$aa^\dagger(a^\dagger a^\dagger|k\rangle) = \psi(k'')(a^\dagger a^\dagger)|k\rangle,$$

etc. Hence, the whole set of eigenstates can be split into a (finite or infinite) set of "bands" of the type

$$\{(a^\dagger)^n|k_1\rangle, \cdots, (a^\dagger)^n|k_\alpha\rangle, \cdots |n \in N_0\}.$$

Remark 5. The procedure can go in the opposite direction:

$$aa^\dagger(a|k\rangle) = a\varphi(k)|k\rangle = \tilde{\psi}'_k a|k\rangle,$$
$$a^\dagger a(a|k\rangle) = \tilde{\varphi}'_k a|k\rangle,$$
$$aa^\dagger(aa|k\rangle) = a\tilde{\varphi}''(k)a|k\rangle = \tilde{\psi}''(k)aa|k\rangle.$$

Hence, the necessary and sufficient condition for the one-dimensional quantum-mechanical problem to allow a Fock-space representation is $[aa^\dagger, a^\dagger a] = 0$, $\langle 0|aa^\dagger|0\rangle > 0$, $(aa^\dagger \neq a^\dagger a)$. If $aa^\dagger = a^\dagger a$, $a|0\rangle = 0$, there is no other state except $|0\rangle$.

3.1 Undeformed (ordinary) quantum-mechanical problems in one dimension

Let us apply the condition for the Fock-space representation (12) to undeformed quantum mechanics, with the following ansatz:

$$\sqrt{2}a = \mathrm{i}p + w(x) = \frac{\mathrm{d}}{\mathrm{d}x} + w(x),$$
$$\sqrt{2}a^\dagger = -\mathrm{i}p + \bar{w}(x) = -\frac{\mathrm{d}}{\mathrm{d}x} + \bar{w}(x). \qquad (15)$$

That gives us

$$2a^\dagger a = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + |w(x)|^2 - \frac{\mathrm{d}}{\mathrm{d}x}w(x) + \bar{w}(x)\frac{\mathrm{d}}{\mathrm{d}x},$$
$$2aa^\dagger = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + |w(x)|^2 + \frac{\mathrm{d}}{\mathrm{d}x}\bar{w}(x) - w(x)\frac{\mathrm{d}}{\mathrm{d}x}. \qquad (16)$$

For the real function w, these equations lead to

$$\left[H_0, \frac{\mathrm{d}w}{\mathrm{d}x}\right] = 0, \ H_0 = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}w^2(x)^2 = \frac{1}{2}\{a, a^\dagger\},$$
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right) - \frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}^2}{\mathrm{d}x^2} = 0,$$
$$w''' + 2w''\frac{\mathrm{d}}{\mathrm{d}x} = 0,$$
$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 0 \implies w = \alpha x + \beta, \qquad (17)$$

where $\alpha, \beta \in \mathbb{R}$.

Next, we consider two special cases with the complex parameters α and β:

(i) $\sqrt{2}a = \frac{\mathrm{d}}{\mathrm{d}x} + \beta,$
$$\sqrt{2}a^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x} + \bar{\beta},$$
$$2a^\dagger a = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + |\beta|^2 - \beta\frac{\mathrm{d}}{\mathrm{d}x} + \bar{\beta}\frac{\mathrm{d}}{\mathrm{d}x},$$
$$H = \frac{1}{2}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + |\beta|^2\right) - \mathrm{i}\Im m\,\beta\frac{\mathrm{d}}{\mathrm{d}x},$$
$$\Re e\,\beta = 0 \implies aa^\dagger = a^\dagger a,$$
$$\Im m\,\beta = 0 \implies aa^\dagger = a^\dagger a,$$
$$2aa^\dagger = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + |\beta|^2 + \bar{\beta}\frac{\mathrm{d}}{\mathrm{d}x} - \beta\frac{\mathrm{d}}{\mathrm{d}x} = 2a^\dagger a. \qquad (18)$$

Since $aa^\dagger = a^\dagger a$, and $a|0\rangle = 0$, there is no Fock-space representation.

(ii) $\sqrt{2}a = \mathrm{i}p + \alpha x = \frac{\mathrm{d}}{\mathrm{d}x} + x\alpha,$
$$\sqrt{2}a^\dagger = -\mathrm{i}p + x\bar{\alpha} = -\frac{\mathrm{d}}{\mathrm{d}x} + x\bar{\alpha},$$
$$2a^\dagger a = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + |\alpha|^2 x^2 - \alpha\frac{\mathrm{d}}{\mathrm{d}x}x + x\bar{\alpha}\frac{\mathrm{d}}{\mathrm{d}x},$$
$$2aa^\dagger = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + |\alpha|^2 x^2 + \bar{\alpha}\frac{\mathrm{d}}{\mathrm{d}x}x - x\alpha\frac{\mathrm{d}}{\mathrm{d}x},$$
$$2a^\dagger a = 2H_0 - \alpha - 2\mathrm{i}\Im m\,\alpha x\frac{\mathrm{d}}{\mathrm{d}x},$$
$$2aa^\dagger = 2H_0 + \bar{\alpha} - 2\mathrm{i}\Im m\,\alpha x\frac{\mathrm{d}}{\mathrm{d}x} = 2a^\dagger a + (\alpha + \bar{\alpha}). \qquad (19)$$

Hence, the Fock-space representation exists if $\alpha \neq 0$. In the limit $\alpha \to 0$, there is no Fock-space representation.

3.2 q-deformed quantum-mechanical problems in one dimension

Let us apply the conditions $[aa^\dagger, a^\dagger a] = 0$, $\langle 0|aa^\dagger|0\rangle \neq 0$ to the operators $a, a^\dagger$ in (11), which satisfy the q-deformed Heisenberg algebras, (1) with $q \in \mathbb{R}$, (satisfying the Leibniz rule). We generalize the above considerations with a natural (linear) restriction:

$$aa^\dagger - u_q a^\dagger a = v_q, \ u_q, v_q \in \mathbb{R}, \qquad (20)$$

and $u_q \to 1$, $v_q \to 1$ when $q \to 1$.

Remark. Generally, it could be the case that $aa^\dagger = \sum_{k=0}^\infty c_k(a^\dagger a)^k$.

Then the conditions $\langle 0|aa^\dagger|0\rangle > 0$ and $[aa^\dagger, a^\dagger a] = 0$ are obviously (automatically) satisfied if $v_q > 0$ (or $c_0 > 0$).

(i) A simple ansatz is

$$a = F_1(N) + F_2(N)p,$$
$$a^\dagger = \bar{F}_1(-N-1) + p\bar{F}_2(-N-1). \qquad (21)$$

Then

$$aa^\dagger = F_1(N)\bar{F}_1(-N-1) + pF_2(N-1)\bar{F}_2(-N-2)p$$
$$+ pF_1(N-1)\bar{F}_2(-N-1) + \bar{F}_1(-N-2)F_2(N)p,$$
$$a^\dagger a = F_1(N)\bar{F}_1(-N-1) + pF_2(N)\bar{F}_2(-N-1)p$$
$$+ pF_1(N)\bar{F}_2(-N-1) + \bar{F}_1(-N-1)F_2(N)p, \qquad (22)$$

In order to obtain relation (20), it is necessary that

$$F_1(N)\bar{F}_1(-N-1) = |\alpha|^2,$$
$$F_2(N)\bar{F}_2(-N-1) = |\beta|^2, \ \alpha, \beta \in \mathbf{C}. \qquad (23)$$

A simple solution for $F_2(N)$ is

$$F_2(N) = \beta q^{N+1/2}, \ q > 1. \qquad (24)$$

Then, in order to cancel the terms with p^2 and p, we obtain

$$F_1(N) = \alpha q^{2(N+1/2)}, \tag{25}$$

since

$$\begin{aligned}
F_2(N-1)\bar{F}_2(-N-2) &= |\beta|^2 q^{(N-1/2)} q^{(-N-3/2)} \\
&= q^{-2} F_2(N)\bar{F}_2(-N-1), \\
F_1(N-1) &= q^{-2} F_1(N). \tag{26}
\end{aligned}$$

Hence the q-deformed algebra is identical with that of [4–6]

$$aa^\dagger - q^{-2} a^\dagger a = \left(1 - q^{-2}\right)|\alpha|^2 = 1, \ q > 1. \tag{27}$$

Note that in the limit $q \to 1$, $\alpha = $ const, we find $aa^\dagger = a^\dagger a$ and there is no Fock-space representation for $q = 1$. The Hamiltonian $H = a^\dagger a$ is given by

$$\begin{aligned}
H &= |\alpha|^2 + |\beta|^2 p^2 + \alpha\bar{\beta} p q^{N+1/2} + \text{h.c.}, \\
H &= \frac{1}{1-q^{-2}} + \frac{1}{2} p^2 + \frac{1}{\sqrt{2(1-q^{-2})}} p q^{N+1/2} + \text{h.c.}.
\end{aligned} \tag{28}$$

The vacuum state $(a|0\rangle_q = 0)$ is given by

$$\left(\alpha q^{2(N+1/2)} + \beta q^{N+1/2} p \right) \sum_{k=0}^\infty c_k x^k = 0,$$

$$\alpha \sum_{k=0}^\infty c_k q^{2(k+1/2)} x^k - i\beta \sum_{k=1}^\infty c_k q^{k-1/2}[k] x^{k-1} = 0, \tag{29}$$

where

$$[k] = f(k) = \frac{q^k - q^{-k}}{q - q^{-1}}.$$

The recursion relations are

$$\begin{aligned}
\alpha c_k q^{2(k+1/2)} - i\beta c_{k+1} q^{k+1/2}[k+1] &= 0, \\
c_{k+1} &= \frac{\alpha}{i\beta} \frac{q^{k+1/2}}{[k+1]} c_k. \tag{30}
\end{aligned}$$

We find that

$$c_k = \left(\frac{\alpha}{i\beta}\right)^k \frac{q^{k/2}}{\{k\}!} c_0, \ \text{where} \ \{k\} = \frac{1 - q^{-2k}}{1 - q^{-2}}. \tag{31}$$

Hence the vacuum is

$$|0\rangle_q = c_0 \sum_{k=0}^\infty \left(-\frac{i\alpha}{\beta} q^{1/2} x\right)^k \frac{1}{\{k\}!} = c_0 \sum_{k=0}^\infty \frac{\left(ip_0 q^{1/2} x\right)^k}{\{k\}!},$$

$$p_0 = -\frac{\alpha}{\beta}. \tag{32}$$

When p is fixed and $q \to 1$, then

$$|0\rangle_q \longrightarrow e^{ip_0 x}.$$

However, in this limit we obtain the free Hamiltonian boosted by $p_0 = -\alpha/\beta$, and $aa^\dagger = a^\dagger a$, so there is no Fock-space representation. Namely, in the limit $q \to 1$, $\alpha = $ const, we obtain

$$(\alpha + \beta p)e^{ip_0 x} = (\alpha + \beta p_0)e^{ip_0 x} = 0 \Longrightarrow p_0 = -\frac{\alpha}{\beta},$$

and there is no Fock-space representation.

Only in the limit $|\alpha| = 1/(1 - q^{-2})^{1/2} \to \infty$, i.e., in the infinite-momentum frame, could one have the ordinary harmonic oscillator picture [4]. We show (see Appendix) that in the infinite-momentum frame:

$$\begin{aligned}
p_0 &= -\frac{\alpha}{\beta} = \pm\sqrt{\frac{2}{1-q^{-2}}}, \\
\lim_{p_0 \to \infty} e^{-ip_0 x}|0\rangle_q &= c_0 e^{-x^2/2}, \\
a &= \pm\frac{1}{\sqrt{2}} x \pm i\frac{p}{\sqrt{2}}, \ m = \omega = 1. \tag{33}
\end{aligned}$$

Generally, if $|\alpha|^2 = 1/(1 - q^{-2})$, $q > 1$, one obtains [4]

$$\begin{aligned}
aa^\dagger &= q^{-2(N_a+1)} E_0 + \frac{1 - q^{-2(N_a+1)}}{1 - q^{-2}}, \\
a^\dagger a &= q^{-2N_a} E_0 + \frac{1 - q^{-2N_a}}{1 - q^{-2}}, \tag{34}
\end{aligned}$$

where

$$[N_a, a] = -a, \ [N_a, a^\dagger] = a^\dagger,$$

$$a^\dagger a|0\rangle = E_0|0\rangle = 0, \ \lim_{n\to\infty} \langle n|a^\dagger a|n\rangle = \frac{1}{1-q^{-2}} > 0.$$

Hence, $E_0 = 0$ defines an ordinary Fock-space representation with the bounded spectrum. Any $E_0 > 1/(1 - q^{-2})$ defines the representation with the unbounded spectrum and with the states $(a^\dagger)^n|E_0\rangle$, $a^n|E_0\rangle$, $n \in N_0$. If $E_0 < 1/(1 - q^{-2})$, the eigenvalues of the states $a^n|E_0\rangle$ would have values unbounded from below for $n \to \infty$. Only if $E_0 = (1 - q^{-2n})/(1 - q^{-2})$, $n \in N_0$, then the lowest eigenvalue is zero. The limit $q = 1$ reduces to the case of ordinary harmonic oscillator with $a^\dagger a = N$.

(ii) In the q-deformed quantum mechanics [10,4] defined by (4) we try to find such a Fock-space representation that leads to the ordinary harmonic oscillator picture in the limit $q \to 1$. Hence, we take a simple ansatz in the following form:

$$\begin{aligned}
a &= f(N) x + ig(N) p, \\
a^\dagger &= x\bar{f}(-N-1) - ip\bar{g}(-N-1), \tag{35}
\end{aligned}$$

where $f(N)$, $g(N)$ are complex functions and for $q \to 1$ these functions tend to $1/2^{1/2}$. We find that

$$\begin{aligned}
a^\dagger a &= x\bar{f}(-N-1)f(N)x + p\bar{g}(-N-1)g(N)p \\
&\quad + ix\bar{f}(-N-1)g(N)p - ip\bar{g}(-N-1)f(N)x, \\
aa^\dagger &= f(N)x^2\bar{f}(-N-1) + g(N)p^2\bar{g}(-N-1) \\
&\quad + ig(N)px\bar{f}(-N-1) - if(N)xp\bar{g}(-N-1). \tag{36}
\end{aligned}$$

A simple analysis shows that in any linear combination of $aa^\dagger$ and $a^\dagger a$ one cannot eliminate all terms on the right-hand side of (36) in such a way as to obtain a constant. Namely, one can eliminate the terms with x^2, p^2, but the terms with xp (or px) remain (unless f, g are constants and $p = -\mathrm{id}/\mathrm{d}x$). Generally, one can simultaneously eliminate only three of the four terms (x^2, p^2, xp, px), and the condition $[aa^\dagger, a^\dagger a] = 0$ is not satisfied. For example, if we take the Hamiltonian with a quadratic potential, we have

$$p = -\mathrm{i}\nabla = \frac{1}{x}\frac{\sinh(hN)}{\sinh(h)},$$
$$\sqrt{2}a = x + \mathrm{i}p, \ \sqrt{2}a^\dagger = x - \mathrm{i}p,$$
$$H = \frac{1}{2}\{a, a^\dagger\}.$$

Since $[aa^\dagger, a^\dagger a] \neq 0$, there is no Fock-space representation. Therefore, in the q-deformed quantum mechanics [10,4] given by (35) there is no deformed harmonic oscillator that goes to $\sqrt{2}a = x+\mathrm{i}p$ in the limit $q = 1$. Having in mind the results of Sect. 3.2, it seems that there is one q-deformed oscillator, (21), that leads to the ordinary harmonic oscillator only in the infinite-momentum frame, (33) (otherwise it leads to the free Hamiltonian without a Fock-space representation). Our conclusion is that for the deformed momentum $p = -\mathrm{i}\nabla$, with ∇ satisfying the Leibniz rule (6), there is no natural generalization of the creation and annihilation operators $a^\dagger$ and a leading to the ordinary harmonic oscillator operators, (33), in the limit $q \to 1$.

4 Calogero model as a deformed Heisenberg algebra with a Fock-space representation

Let us now investigate the structure of the ∇ operator in order to get the deformed Heisenberg algebra with a Fock-space representation. Assuming that the creation and annihilation operators depend linearly on x and ∇ (as in the harmonic oscillator model),

$$a = \frac{1}{\sqrt{2}}(\nabla + x),$$
$$a^\dagger = \frac{1}{\sqrt{2}}(-\nabla + x), \tag{37}$$

and insisting on the consistency condition (12), we easily obtain

$$[-\nabla^2 + x^2, f(N + 1) - f(N)] = 0, \tag{38}$$

where the function f has been introduced in (5). Consequently, the function f is an identity

$$f(N) = N, \tag{39}$$

or satisfies the condition

$$f(N + 3) - f(N + 2) = f(N + 1) - f(N). \tag{40}$$

The first possibility leads to the well-known case of undeformed harmonic oscillator, while the second one gives

$$f(N) = N + h\sin(\pi N). \tag{41}$$

The ∇ operator can now be expressed as

$$\nabla = \partial + \frac{h}{x}\sin(\pi N). \tag{42}$$

Note that the ∇ operator does not satisfy the generalized Leibniz rule (6). The annihilation and creation operators can be rewritten as

$$a = \frac{1}{\sqrt{2}}\left(\partial + \frac{h}{x}\sin(\pi N) + x\right),$$
$$a^\dagger = \frac{1}{\sqrt{2}}\left(-\partial - \frac{h}{x}\sin(\pi N) + x\right). \tag{43}$$

The Hamiltonian can now be expressed as

$$H = a^\dagger a = -\frac{1}{2}\partial^2 + \frac{x^2}{2} - \frac{1}{2} + h\sin(\pi N)$$
$$+ \frac{h}{2x^2}\sin(\pi N) + \frac{h^2}{2x^2}\sin^2(\pi N). \tag{44}$$

This is in fact the Calogero two-body Hamiltonian [16] with the harmonic potential, where x denotes the relative coordinate $x = x_1 - x_2$. Namely, the action of the $\sin(\pi N)$ operator on the single power of x, say x^λ, reduces to

$$\sin(\pi N)x^\lambda = \sin(\pi\lambda)x^\lambda. \tag{45}$$

It can be shown that the eigenfunctions of the Hamiltonian are given by

$$\psi_n(x) = x^{\nu+n}\mathrm{e}^{-x^2/2} = (a^\dagger)^n\psi_0(x) \tag{46}$$

and the corresponding eigenvalues by

$$\epsilon_n = n + \nu[1 - (-)^n], \tag{47}$$

where $\nu = -h\sin(\pi\nu)$ and n is an integer, $n \geq 0$ [17]. This gives for the function f:

$$f(N) = x\nabla = N + h\sin(\pi N) = N - \nu\frac{\sin(\pi N)}{\sin(\pi\nu)}. \tag{48}$$

By acting on the eigenfunctions (46), we can further simplify (48) to

$$x\nabla = N - \frac{\nu\sin(\pi(\nu+n))}{\sin(\pi\nu)} = N - \nu(-)^n = N - \nu K, \tag{49}$$

where K denotes the so-called exchange operator, which, in our case, reduces to the sign flip.

Let us now generalize the above results to the case of N particles. We shall postulate that the N-particle version of the deformed derivation (49) is given by

$$x_{ij}\nabla_{ij} = N_{ij} - \nu K_{ij} \tag{50}$$

for each pair of indices, where

$$x_{ij} = x_i - x_j,$$
$$N_{ij} = x_{ij}\frac{\partial}{\partial x_{ij}}. \tag{51}$$

Here, K_{ij} denotes the particle permutation operator, obeying

$$
\begin{aligned}
K_{ij}x_{ij} &= -x_{ij}K_{ij}, \\
K_{ij} &= K_{ji}, \\
K_{ij}K_{jl} = K_{jl}K_{li} &= K_{li}K_{ij}, \\
K_{ij}^2 &= 1.
\end{aligned}
\tag{52}
$$

The operators ∇_{ij} satisfy the following commutation relations:

$$
\begin{aligned}
[\nabla_{ij}, x_{ik}] &= \delta_{jk}(1 + \nu K_{ij}) + \nu K_{ij}, \\
[\nabla_{ij}, \nabla_{ik}] &= \frac{\nu}{x_{ij}} K_{ij} \left(\frac{\partial}{\partial x_{jk}} - \frac{\partial}{\partial x_{ik}} \right) \\
&+ \frac{\nu}{x_{ik}} K_{ik} \left(\frac{\partial}{\partial x_{ij}} - \frac{\partial}{\partial x_{kj}} \right) \\
&+ \nu^2 \left(\frac{1}{x_{ij}x_{jk}} - \frac{1}{x_{ik}x_{kj}} \right) K_{ij}K_{ik}.
\end{aligned}
\tag{53}
$$

Other commutators (i.e., commutators between the objects with pairwise noncoinciding indices) can be calculated from the set given above (53). Furthermore, we define that the covariant derivative with respect to the ith particle is given by

$$
\nabla_i = \sum_{j \neq i}^{N} \nabla_{ij}.
\tag{54}
$$

The transition from the original set of variables $\{x_i\}$ to the new set of independent variables $\{x_{ij}, X\}$, where X denotes the center-of-mass coordinate,

$$
X = \frac{1}{N} \sum_i^{N} x_i,
\tag{55}
$$

enables us to rewrite the sum (54) as

$$
\nabla_i = \sum_{j \neq i}^{N} \nabla_{ij} = \frac{\partial}{\partial x_i} - \frac{1}{N} \frac{\partial}{\partial X} - \nu \sum_{j \neq i} \frac{1}{x_{ij}} K_{ij}.
\tag{56}
$$

In order to get rid of the center-of-mass degree of freedom, we finally redefine the covariant derivative (54) as

$$
D_i = \nabla_i + \frac{1}{N} \frac{\partial}{\partial X}.
\tag{57}
$$

Having in mind (53), it is easy to see that the covariant derivation D_i and the coordinate x_i satisfy the commutation relations

$$
\begin{aligned}
[D_i, D_j] &= 0, \\
[x_i, x_j] &= 0, \\
[D_i, x_j] &= \delta_{ij} \left(1 + \nu \sum_{l=1}^{N} K_{il} \right) - \nu K_{ij},
\end{aligned}
\tag{58}
$$

identical with those given in the paper [18].

The creation and annihilation operators can be constructed as

$$
\begin{aligned}
a_i^\dagger &= \frac{1}{\sqrt{2}}(-D_i + x_i), \\
a_i &= \frac{1}{\sqrt{2}}(D_i + x_i).
\end{aligned}
\tag{59}
$$

The Hamiltonian can now be expressed as

$$
H = \sum_{i=1}^{N} a_i^\dagger a_i.
\tag{60}
$$

From a series of algebraic manipulations, using the properties of the particle permutation operator K_{ij} (52), the Hamiltonian (60) takes the form

$$
\begin{aligned}
H &= \frac{1}{2} \sum_{i=1}^{N} (-D_i^2 + x_i^2) - \frac{1}{2} \sum_{i=1}^{N} [D_i, x_i] \\
&= -\frac{1}{2} \sum_{i=1}^{N} \partial_i^2 + \frac{1}{2} \sum_{i=1}^{N} x_i^2 + \frac{1}{2} \sum_{i \neq j}^{N} \frac{\nu(\nu - K_{ij})}{(x_i - x_j)^2} \\
&\quad - \frac{\nu^2}{2} \sum_{i \neq j \neq l}^{N} \frac{1}{(x_i - x_j)(x_j - x_l)} K_{ij}K_{jl} \\
&\quad - \frac{1}{2} \sum_{i=1}^{N} [D_i, x_i].
\end{aligned}
\tag{61}
$$

Finally, using the identity

$$
\sum_{i \neq j \neq l}^{N} \frac{1}{(x_i - x_j)(x_j - x_l)} = 0
\tag{62}
$$

and the commutation relation (58), we obtain the Hamiltonian (61) in the form

$$
\begin{aligned}
H &= -\frac{1}{2} \sum_{i=1}^{N} \partial_i^2 + \frac{1}{2} \sum_{i=1}^{N} x_i^2 + + \frac{1}{2} \sum_{i \neq j}^{N} \frac{\nu(\nu - K_{ij})}{(x_i - x_j)^2} \\
&\quad - \frac{1}{2} \sum_{i=1}^{N} \left(1 + \nu \sum_{j \neq i}^{N} K_{ij} \right).
\end{aligned}
\tag{63}
$$

Noting that the operator K_{ij} gives 1 and (-1) in the bosonic and fermionic subspace, respectively, we finally obtain

$$
\begin{aligned}
H &= -\frac{1}{2} \sum_{i=1}^{N} \partial_i^2 + \frac{1}{2} \sum_{i=1}^{N} x_i^2 + \frac{1}{2} \sum_{i \neq j}^{N} \frac{\nu(\nu \mp 1)}{(x_i - x_j)^2} \\
&\quad - \frac{N}{2} \mp \frac{\nu}{2} N(N-1),
\end{aligned}
\tag{64}
$$

where the upper and lower sign refers to the symmetric and antisymmetric functions, respectively. This is simply the Calogero model for N particles in the harmonic potential with the frequency ω equal to 1 [19].

5 Conclusion

We have generalized the q-deformed one-dimensional Heisenberg algebra [2–4,10–13], to arbitrary deformations. We have discussed the conditions leading to a generalized Leibniz rule. We have found two classes of solutions: one already studied by Wess et al. [2–4,10–12] and the other for $q = \exp(ih)$, $h \in \mathbb{R}$. Furthermore, we have given a simple condition that the one-dimensional problem should have a Fock-space representation. In ordinary quantum mechanics, this condition leads to the ordinary harmonic oscillator. We have applied it to the special deformation $\nabla = \sinh(hN)/x\sinh(h)$, $N = xd/dx$, considered in [2–4,10–13]. We have found only one solution leading to a q-deformed oscillator [4], which reduces to the harmonic oscillator only in the infinite-momentum frame.

Finally, we have imposed the condition for a Fock-space representation on $a \sim \nabla + x$, $a^\dagger \sim -\nabla + x$ and found that the solution for ∇ is $\sim N + h\sin(\pi N)$, $N = xd/dx$. This leads to the two-particle Calogero model in ordinary quantum mechanics. We have also found the generalization of the N-particle Calogero model in ordinary quantum mechanics. It is interesting that we have found no other example of deformed quantum mechanics with a Fock-space representation. It would be interesting to find the corresponding two-dimensional oscillator in the quantum plane.

A Appendix

Expanding the plain wave in power series and using the structure of the vacuum (32), we have

$$e^{-ipx}|0\rangle_q = c_0 \sum_{n=0}^{\infty} (-i)^n p_0^n x^n \sum_{k=0}^{n} \frac{(-)^k}{(n-k)!\{k\}!} q^{k/2}. \quad (65)$$

To control the infinite momentum p_0, we expand it around the undeformed point $q = 1+h$, with h as an infinitesimal small quantity. The momentum p_0 goes to infinity as

$$p = -\frac{1}{h^{1/2}}.$$

To proceed, we have to expand the sum

$$f_n(q) = \sum_{k=0}^{n} \frac{(-)^k}{(n-k)!\{k\}!} q^{k/2}$$

into a power series of $h = q - 1$:

$$f_n(q) = \sum_{m=0}^{\infty} \frac{h^m}{m!} f_n^{(m)}(q)\bigg|_{q=1}.$$

It is easy to see that the mth derivative of the function $f(q)$ with respect to q is given by

$$f_n^{(m)}(1) = \sum_{k=0}^{n} \frac{(-)^k}{(n-k)!k!} P_{2m}(k),$$

where $P_{2m}(k)$ is some polynomial in k of order $2m$. Using the identity [20]

$$\sum_{k=0}^{n} \frac{(-)^k}{(n-k)!k!} S_m^{(k)} = 0, \text{ for } n > m,$$

we find that $f_n^{(m)}(1)$ vanishes for $m < n/2$ and n even, and it is different from zero for $m \geq n/2$. For n odd, say $n = 2l + 1$, $f_n^{(m)}(1)$ vanishes for $m \leq l$ and the first nonvanishing value appears for $m = l + 1$. Consequently, $e^{-ip_0 x}|0\rangle_q$ is given by

$$e^{-ip_0 x}|0\rangle_q = c_0 \sum_{n=0}^{\infty} (i)^n h^{-n/2} x^n \sum_{m=0}^{\infty} \frac{h^m}{m!} f_n^{(m)}(1)$$

$$= c_0 \sum_{n=0}^{\infty} (i)^n x^n \sum_{m=0}^{\infty} \frac{h^{m-n/2}}{m!} f_n^{(m)}(1). \quad (66)$$

For n odd, this expansion goes to zero as $h^{1/2}$, while for n even, it reduces to

$$c_0 \sum_{n=0}^{\infty} (i)^n x^n \frac{f_n^{(n/2)}(1)}{\left(\frac{n}{2}\right)!}. \quad (67)$$

Since $\{r\}_{q=1}^{(m)}$ goes like r^{m+1}, the dominant term of $f_n^{(m)}$ is given by

$$\left(\sum_{r=1}^{k} \frac{\{r\}'}{\{r\}}\right)^m \frac{q^{k/2}}{\{k\}!}, \ m \leq k \leq n. \quad (68)$$

One easily finds that the sum over r in (68) is given by

$$\sum_{r=1}^{k} \frac{\{r\}'}{\{r\}} = -\frac{k(k-1)}{2}.$$

Hence, it follows that

$$f_n^{(n/2)}(1) = \frac{1}{2^{n/2}}. \quad (69)$$

Substitution of the expression (69) into (67) results in the Taylor expansion for $\exp(-x^2/2)$.

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